Math 4303 Outline: The Set of Quaternions

We saw in an earlier worksheet that a certain set of real 2×2 matrices with the usual matrix addition and multiplication can be identified with the field of complex numbers. The purpose of this outline is to establish that a certain set of complex 2×2 matrices can serve as a model for the set of quaternions. The quaternions form an associative division algebra (a fancy name for a "field" without commutativity of multiplication).

Let

$$\mathbb{H} = \left\{ \begin{pmatrix} w & z \\ -\overline{z} & \overline{w} \end{pmatrix} \mid w, z \in \mathbb{C} \right\}$$

1. Show: If A and B are elements in $\mathbb{H}$, then $A + B$ is an element of $\mathbb{H}$.
2. Show: If A and B are elements in $\mathbb{H}$, then $A \cdot B$ is an element of $\mathbb{H}$.
3. Show: (1) The determinant of a matrix in $\mathbb{H}$ is a non-negative real number. (2) The zero matrix is the only matrix in $\mathbb{H}$ with determinant 0.
4. Check that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{H}$, and show: If $A \in \mathbb{H}$ is not the zero matrix, then A has a multiplicative inverse in $\mathbb{H}$.
5. Let $w = a + di$ and $z = b + ci$, where $a, b, c, d \in \mathbb{R}$. Find the four matrices $E, I, J, K \in \mathbb{H}$ such that

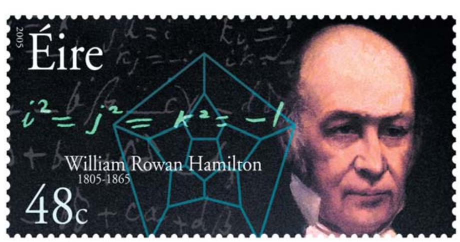
   $$\begin{pmatrix} w & z \\ -\overline{z} & \overline{w} \end{pmatrix} = aE + bI + cJ + dK.$$

   Since the four matrices E, I, J, K form a linearly independent set, $\mathbb{H}$ is a four-dimensional real vector space, and we can write

   $$\mathbb{H} = \{aE + bI + cJ + dK \mid a, b, c, d \in \mathbb{R}\}.$$

6. Show that $\mathbb{H}$ is not closed under multiplication by a complex scalar. (Thus $\mathbb{H}$ is not a complex vector space.)
7. Show that $I \cdot I = J \cdot J = K \cdot K = -E$.
8. Show that $J \cdot K = -K \cdot J = I$. Similarly one can show that $I \cdot J = -J \cdot I = K$ and $K \cdot I = -I \cdot K = J$. (Thus matrix multiplication is not commutative in $\mathbb{H}$.)
9. Compute $(5E - I + 2J + K)(E + 2I - 3J - K)$ in two ways: (1) By using the rules in 7. and 8. assuming the distributive law holds, and (2) by computing the matrix product of the the two matrices in $\mathbb{H}$ corresponding to the two factors above. (This illustrates the validity of the distributive law for quaternions.)
10. Show that $\det(aE + bI + cJ + dK) = a^2 + b^2 + c^2 + d^2$.
11. Given a matrix $A = aE + bI + cJ + dK$, we let $\overline{A} = aE - bI - cJ - dK$. $\overline{A}$ is called the *quaternian conjugate* of A. Show that $A \cdot \overline{A} = \overline{A} \cdot A = \det(A)E$. (It follows that if $\det(A) \neq 0$, then $A^{-1} = \dfrac{\overline{A}}{det(A)}$.)

The quaternions were discovered by William Rowan Hamilton in 1843. According to the Theorem of Frobenius, there are only three real vector spaces that form associative division algebras: $\mathbb{R}$ (one-dimensional), $\mathbb{C}$ (two-dimensional), and $\mathbb{H}$ (four-dimensional).



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