Transformation-based Geometry Through Complex Numbers: Teaching a Capstone Course for Future High School Teachers

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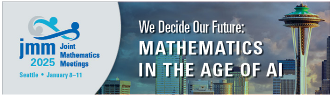
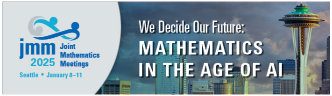


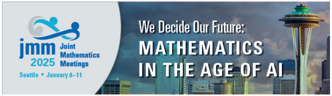
Overview

1. UTEP
2. Course Design and Contents
3. Conceptual Framework
4. Complex Numbers and Geometry



UTEP and Its Students

- 25,000+ students total
- $\sim$ 3,700 graduate students



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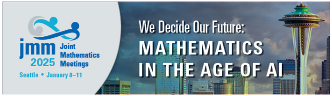
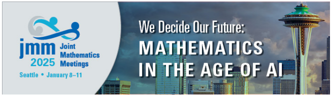


Math 4303
Fundamental Mathematics from an Advanced Standpoint

- The only course in our department taken by future secondary Mathematics teachers that specifically addresses High School Mathematics.



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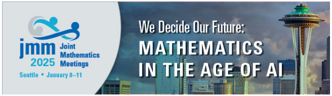

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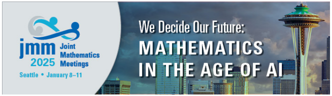
- The prerequisite is an *Introduction to Proof* class.
- The course is almost exclusively taken by juniors and seniors.
- The vast majority of students will be student-teaching within a year.

Course Contents

- From the counting numbers to the real numbers
- **Complex numbers**
- Functions
- Algebra (= Solving Equations)



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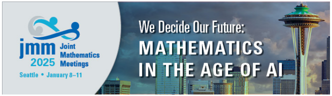

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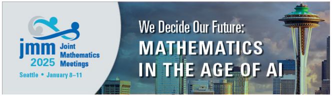

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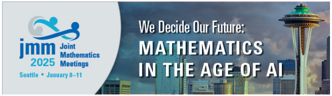
  - $\mathbb{C}$ is also another field of numbers, albeit without a nice order like $\mathbb{R}$



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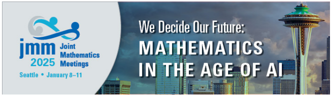

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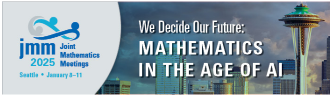
  - $\mathbb{C}$ is algebraically closed
  - The complex plane



Basic Characteristics of a "Working" Mathematics Classroom by H. Wu

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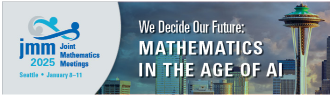
1. **Precision:** Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.
2. **Definitions:** They are the bedrock of the mathematical structure. They are the platform that supports reasoning. No definitions, no mathematics.

Basic Characteristics of a Mathematics Classroom by H. Wu

4. **Reasoning:** The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching—and learning—by rote.
5. **Purposefulness:** Mathematics is goal-oriented, and every concept or skill is there for a purpose.
6. **Coherence:** Mathematics is a tapestry in which all the concepts and skills are interwoven. It is all of a piece.



An Example of Coherence:
The First Baby Step of the Erlangen Program

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is called an *isometry* if it satisfies

$$|f(w) - f(z)| = |w - z| \text{ for all } w, z \in \mathbb{C}.$$

Examples. ???

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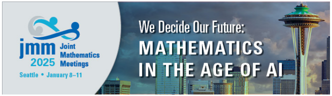
Examples. Translations, ???, Rotations, Reflections.



Erlangen Program II

Theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary isometry. Then there exist $z_0 \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $f(z) = z_0 + ze^{i\theta}$ or $f(z) = z_0 + \overline{z}e^{i\theta}$.

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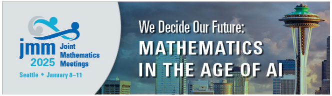
Lemma. Let $h : \mathbb{C} \to \mathbb{C}$ be an isometry that satisfies $h(0) = 0$ and $h(1) = 1$. Then $h(z) = z$ or $h(z) = \bar{z}$.



Wu’s Principles and Student Beliefs:
2. Definitions

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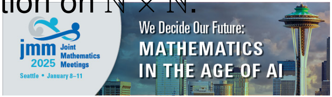
- Recall that a *relation on a set X* is a subset of the Cartesian product $X \times X$.

  We define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ as follows:

  $$(p, q) \sim (p', q') \Leftrightarrow p + q' = p' + q.$$

  Show that $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.



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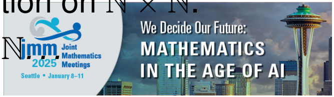
No student group notices that $X = \mathbb{N} \times \mathbb{N}$.



Wu's Principles and Student Beliefs:
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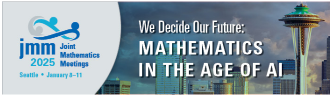
- Based on their own HS experience(?), students will not expect mathematical reasoning from their pupils.



Wu's Principles and Student Beliefs:
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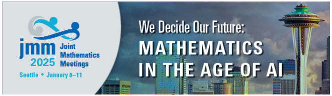
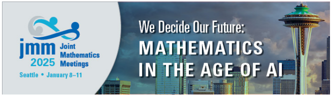
- Based on their own HS experience(?), students will not expect mathematical reasoning from their pupils.
- Students need to understand that reasoning with definitions will be the only way for them to "check" the Mathematics they will be teaching.



Example: Discussion of a Test Problem (after finishing the Equations chapter)

- Solve $\log_2 x - \log_2(1 + \sqrt{x}) = -1$.



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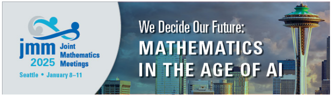

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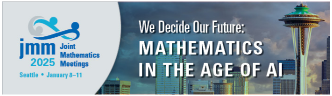

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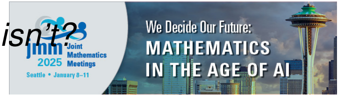
Why is $x = 1$ a solution, while $x = 1/4$ isn't?



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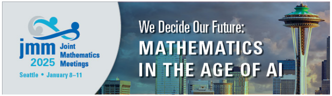

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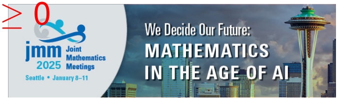
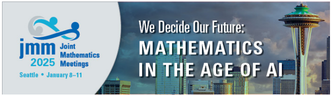
- $\Leftrightarrow 4x^2 - 4x + 1 = x \quad \wedge \quad 2x - 1 \geq 0$
- $\Leftrightarrow (4x-1)(x-1) = 0 \quad \wedge \quad 2x - 1 \geq 0$
- $\Leftrightarrow x = 1$

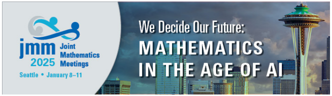


Wu’s Principles and Student Beliefs:
4. Purposefulness

- Intrinsic motivation seems to be a completely new concept to students.
- Their HS and college experience for the most part seem to have been mathematically “purposeless”.



Example: Questions During the Trial Run



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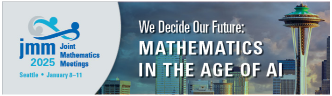
- What is the purpose of the material you present?



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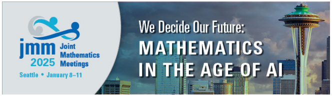

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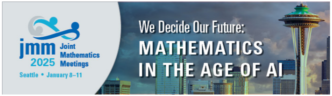
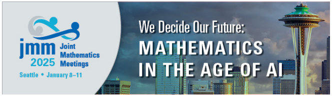
- What is the purpose of the material you present?
- Does the material you present tell a story?
- Why do you present this example? What purpose does it serve?



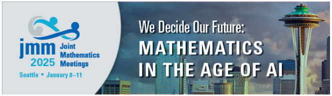
Wu's Principles and Student Beliefs:
5. Coherence

- Students do not seem to have been exposed much to the idea that "Mathematics is a tapestry in which all the concepts and skills are interwoven".
- Mathematics seems to have been taught to them as a collection of more or less unrelated mathematical subjects.



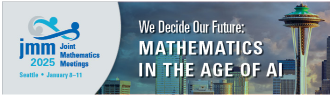
Topic: Algebra

- Being essentially an Algebra course, the course may contribute to the problem.
- The connection between “Modern Algebra” and “HS Algebra” is stressed.



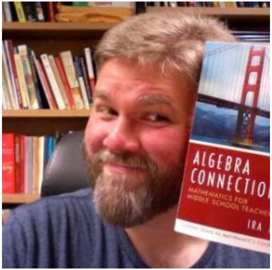
A Link between Complex Numbers and Geometry: Classification of Isometries in the Complex Plane

- **Theorem.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an arbitrary isometry. Then there exist* $z_0 \in \mathbb{C}$ *and* $\theta \in \mathbb{R}$ *such that* $f(z) = z_0 + ze^{i\theta}$ *or* $f(z) = z_0 + \overline{z}e^{i\theta}$.

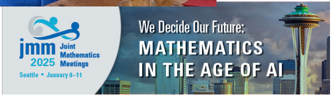


Credits

Matthew Winsor



Heinz Althoff



References

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