Reflective thinking turns experience into insight.
John Maxwell

1 Numbers

In 1879, Gottlob Frege completed the first step of his program to put mathematics on a solid foundation. His idea was that **logic** should be the foundation of all mathematics, and, following Gottfried von Leibniz (1646–1716) and George Boole (1815–1864), he created a rigorous symbolic language, which he called *Begriffsschrift*, to incorporate all standard principles of logic.

Georg Cantor followed in his footsteps and developed **set theory** from basic logical principles. In 1888, Richard Dedekind took the next step, and presented a **construction of the real numbers** based on set theory.

It should be mentioned that Frege's program was doomed to fail. Frege's construction allowed objects such as "the set of all sets". Bertrand Russell used this to construct a paradox: Let E denote the set of all sets which do not contain themselves as members. Is E an element of E? It can't be, because E contains only sets which are **not** members of themselves. Can E fail to be an element of E? No, since if $E \notin E$, then by the definition of the set E, E is contained in E.

Bertrand Russell's and Alfred Whitehead's attempts to "fix" these problems in their monumental *Principia Mathematica* are generally regarded as artificial and therefore in violation of the spirit of Frege's program.

In response, David Hilbert came up with an alternative program: Use axiomatic systems as the foundation of mathematics together with *meta-mathematics*. Mathematicians "do" mathematics starting from axiomatic systems; meta-mathematics allows to talk about the process "from the outside" addressing issues such as completeness[1] and consistency[2] of a given axiomatic system.

In 1930, Kurt Gödel showed that this approach was equally flawed: It is not possible to show (within the axiomatic system) that an axiomatic system which incorporates the arithmetic of natural numbers is complete (or consistent).

[1] An axiomatic system is complete, if all statements within the axiomatic system can—in principle—be shown to be true or to be false.

[2] An axiomatic system is said to be consistent, if the axioms can be shown not to lead to contradictions.

1.1 The Natural Numbers

Definition. Richard Dedekind started by giving the following definition of the set of **Natural Numbers**[3]:

> *The natural numbers are a set* $\mathbb{N}$ *containing a special element called* 0*, and a function* $S : \mathbb{N} \to \mathbb{N}$ *satisfying the following axioms:*
>
> **(D1)** S *is injective*[4].
>
> **(D2)** $S(\mathbb{N}) = \mathbb{N} \setminus \{0\}$.[5]
>
> **(D3)** *If a subset* M *of* $\mathbb{N}$ *contains 0 and satisfies* $S(M) \subseteq M$*, then* $M = \mathbb{N}$.

The function S is called the successor function.

The first two axioms describe the process of counting, the third axiom assures the **Principle of Induction**:

Task 1.1

Let $P(n)$ be a predicate with the set of natural numbers as its domain. If

1. $P(0)$ is true, and
2. $P(S(n))$ is true, whenever $P(n)$ is true,

then $P(n)$ is true for all natural numbers.

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[3]A similar definition of the natural numbers was introduced by Giuseppe Peano in 1889:

> *The natural numbers are a set* $\mathbb{N}$ *containing a special element called* 0*, and a function* $S : \mathbb{N} \to \mathbb{N}$ *satisfying the following axioms:*
>
> **(P1)** $0 \in \mathbb{N}$.
>
> **(P2)** *If* $n \in \mathbb{N}$*, then* $S(n) \in \mathbb{N}$.
>
> **(P3)** *If* $n \in \mathbb{N}$*, then* $S(n) \neq 0$.
>
> **(P4)** *If a set* A *contains* 0*, and if* A *contains* $S(n)$*, whenever it contains* n*, then the set* A *contains* $\mathbb{N}$.
>
> **(P5)** $S(m) = S(n)$ *implies* $m = n$ *for all* $m, n \in \mathbb{N}$.

[4]A function $f : A \to B$ is called *injective* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

[5]For a function $f : A \to B$, $f(A) := \{b \in B \mid f(a) = b \text{ for some } a \in A\}$.

Arithmetic Properties. **Addition** of natural numbers is established recursively in the following way: For a fixed but arbitrary $m \in \mathbb{N}$ we define

$$\begin{aligned} m + 0 &:= m \\ m + S(n) &:= S(m+n) \qquad \text{for all } n \in \mathbb{N} \end{aligned}$$

By Axiom (D3), adding n to the fixed m is then defined for all natural numbers n. It is not clear at this point that the recursive formula defines addition in a unique way. This will be proved later in Task 1.21.

Task 1.2
If we set $S(0) := 1$, then $S(m) = m + 1$ for all natural numbers $m \in \mathbb{N}$.

Use induction for the following:

Task 1.3
Show that addition on $\mathbb{N}$ is associative.

Task 1.4
Show that addition on $\mathbb{N}$ is commutative.

This last task implies in particular that 0 is the (unique) neutral element with respect to addition: $n + 0 = 0 + n = n$ holds for all $n \in \mathbb{N}$.

Here is the cancellation law for addition:

Task 1.5
If $m + k = n + k$, then $m = n$.

Multiplication of natural numbers is also defined recursively as follows: For a fixed but arbitrary $m \in \mathbb{N}$ we define

$$\begin{aligned} m \cdot 0 &:= 0 \\ m \cdot (n+1) &:= m \cdot n + m \qquad \text{for all } n \in \mathbb{N} \end{aligned}$$

Task 1.22 will show that this recursive formula defines multiplication in a unique manner.

Task 1.6
Show that the following distributive law holds for natural numbers:

$$(m + n) \cdot k = m \cdot k + n \cdot k.$$

Task 1.7
Show that 1 is the neutral element with respect to multiplication: For all natural numbers m,

$$m \cdot 1 = 1 \cdot m = m.$$

Task 1.8
Show that multiplication on $\mathbb{N}$ is commutative.

Task 1.9
Show that multiplication on $\mathbb{N}$ is associative.

Task 1.10
Show that multiplication is zero-divisor free:

$$m \cdot n = 0 \text{ implies } m = 0 \text{ or } n = 0.$$

Finally we can impose a **total order**[6] on $\mathbb{N}$ as follows: We say that $m \leq n$, if there is a natural number k, such that $m + k = n$.

Show that "$\leq$" is indeed a total order:

Task 1.11
"$\leq$" is reflexive[7].

Task 1.12
"$\leq$" is anti-symmetric[8].

[6] A relation $\sim$ on A is called a *total order*, if $\sim$ is reflexive, anti-symmetric, transitive, and has the property that for all $a, b \in A$, $a \sim b$ or $b \sim a$ holds.

[7] A relation $\sim$ on A is *reflexive* if for all $a \in A$, $a \sim a$.

[8] A relation $\sim$ on A is *anti-symmetric* if for all $a, b \in A$ the following holds: $a \sim b$ and $b \sim a$ implies that a=b.

Task 1.13
"$\leq$" is transitive[9].

Task 1.14
For all $m, n \in \mathbb{N}$, $m \leq n$ or $n \leq m$.

Show the following two compatibility laws:

Task 1.15
If $m \leq n$, then $m + k \leq n + k$ for all $k \in \mathbb{N}$.

Task 1.16
If $m \leq n$, then $m \cdot k \leq n \cdot k$ for all $k \in \mathbb{N}$.

Last not least, here is the cancellation law for multiplication:

[9] A relation $\sim$ on A is *transitive* if for all $a, b, c \in A$ the following holds: $a \sim b$ and $b \sim c$ implies that $a \sim c$.

Task 1.17
If $m \cdot k = n \cdot k$, then $m = n$ or $k = 0$.

Infinite Sets and the Existence of the Set of Natural Numbers. Do natural numbers exist? Following Dedekind, we will say that a set M is **infinite**, if there is an injective map $f : M \to M$ that is not surjective[10].

Task 1.18
Show that the set of natural numbers as defined on p. 2 is infinite.

Thus, the existence of the set of natural numbers implies the existence of infinite sets. In fact, we will show that the converse also holds:

Theorem. If there is an infinite set, then there is a model for the natural numbers.

Proof: Let A be an infinite set. Then there is a function $S : A \to A$ that is injective, but not surjective. Thus we can find an $a_0 \in A$ with $a_0 \notin S(A)$. Let

$$\mathcal{K} = \{B \subseteq A \mid a_0 \in B \text{ and } S(B) \subseteq B\}$$

Note that $A \in \mathcal{K}$, so $\mathcal{K} \neq \emptyset$. We set

$$N = \bigcap_{B \in \mathcal{K}} B.$$

Observe that $N \in \mathcal{K}$. Indeed, $a_0 \in N$, since $a_0 \in B$ for all $B \in \mathcal{K}$. Also

$$S(N) = S\left(\bigcap_{B \in \mathcal{K}} B\right) \subseteq \bigcap_{B \in \mathcal{K}} S(B) \subseteq \bigcap_{B \in \mathcal{K}} B = N.$$

[10] A function $f : A \to B$ is called *surjective*, if $f(A) = B$.

By its definition the set N is thus the smallest element of $\mathcal{K}$.

Finally we show that N with the function $S : N \to N$ (as successor function) and a_0 (in the role of 0) satisfies Axioms (D1)–(D3).

As the restriction of the injective function $S : A \to A$ to N, the function $S : N \to N$ is also injective. Thus (D1) is satisfied.

For (D2) we have to show that $S(N) = N \setminus \{a_0\}$. Since $a_0 \notin S(N)$ and $S(N) \subseteq N$, we obtain that $S(N) \subseteq N \setminus \{a_0\}$. For the remaining subset relation suppose to the contrary that there is a second element missing from the range of N: there is an element $n_0 \in N$ satisfying $n_0 \notin S(N)$ and $n_0 \neq a_0$. Set $N_0 = N \setminus \{n_0\}$. Note that $a_0 \in N_0$ and that $S(N_0) \subseteq N_0$. Thus $N_0 \in \mathcal{K}$. We also know that $N_0 \subsetneqq N$, yielding a contradiction.

Now let $M \subseteq N$, with $a_0 \in M$, and satisfying $S(M) \subseteq M$. Then $M \in \mathcal{K}$, and thus, again using the minimality of N in $\mathcal{K}$, it follows that $M \supseteq N$. This proves (D3) and completes the proof.

Task 1.19
Present the proof of this Theorem.

Recursion and Uniqueness. Before we give a proof of the "essential" uniqueness of the natural numbers, we will follow Dedekind and establish the following general **Recursion Principle**:

Task 1.20
Let A be an arbitrary set, and let $a \in A$ and a function $f : A \to A$ be given. Then there exists a unique map $\varphi : \mathbb{N} \to A$ satisfying

1. $\varphi(0) = a$, and
2. $\varphi \circ S = f \circ \varphi$.

Here is a possible outline for a proof: Consider all subsets $K \subseteq \mathbb{N} \times A$ with the following properties:

1. $(0, a) \in K$, and
2. If $(n, b) \in K$, then $(S(n), f(b)) \in K$.

Clearly $\mathbb{N} \times A$ itself has these properties; we can therefore define the smallest such set: Let

$$L = \bigcap \{K \subseteq \mathbb{N} \times A \mid K \text{ satisfies (1) and (2)}\}.$$

Now show by induction that for every $n \in \mathbb{N}$ there is a unique $b \in A$ with $(n, b) \in L$. This property defines φ by setting $\varphi(n) = b$ for all $n \in \mathbb{N}$.

The Recursion Principle makes it possible to define a recursive procedure (the function φ) via a formula (the function f).

Task 1.21
Define addition of an arbitrary natural number n and the fixed natural number m using the Recursion Principle.

Task 1.22
Define multiplication of an arbitrary natural number n with the fixed natural number m using the Recursion Principle.

Use the Recursion Principle to show that the set of natural numbers is unique in the following sense:

Task 1.23
Suppose that $\mathbb{N}$, $S : \mathbb{N} \to \mathbb{N}$ and 0 satisfy Axioms (D1)–(D3), and that $\mathbb{N}'$, $S' : \mathbb{N}' \to \mathbb{N}'$ and $0'$ satisfy Axioms (D1)–(D3) as well.

Then there is a bijection[11] $\varphi : \mathbb{N} \to \mathbb{N}'$ such that

1. $\varphi(0) = 0'$, and
2. $\varphi \circ S = S' \circ \varphi$.

[11] A function $f : A \to B$ is a *bijection*, if it is both injective and surjective.

1.2 The Integers

Definition. Integers can be written as differences of natural numbers. The set of integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ will therefore be defined as certain equivalence classes of the two-fold Cartesian product of $\mathbb{N}$.

We define a relation on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c.$$

The next three tasks show that "$\sim$" defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$:

Task 1.24

1. "$\sim$" is reflexive.
2. "$\sim$" is symmetric[12].

Task 1.25

"$\sim$" is transitive.

We will denote equivalence classes as follows:

$$(a, b)_\sim := \{(c, d) \mid (c, d) \sim (a, b)\}.$$

The set of integers $\mathbb{Z}$ is the set of all equivalence classes obtained in this manner:

$$\mathbb{Z} = \{(a, b)_\sim \mid a, b \in \mathbb{N}\}.$$

Addition of integers will be defined component-wise:

$$(a, b)_\sim + (c, d)_\sim = (a + c, b + d)_\sim.$$

[12] A relation $\sim$ on A is called *symmetric*, if for all $a, b \in A$ the following holds: $a \sim b$ implies $b \sim a$.

A set G with a binary operation $\star$ is called an *Abelian group* if $\star$ is commutative and associative, if $(A, \star)$ has a neutral element n satisfying $g \star n = g$ for all $g \in G$, and if $(A, \star)$ has inverse elements, i.e., for all $g \in G$ there is an $h \in G$ satisfying $g \star h = n$.

The next five tasks will show that $\mathbb{Z}$ is an Abelian group with respect to addition.

Task 1.26
Show that the addition of integers is well-defined (i.e. independent of the chosen representatives of the equivalence classes).

Task 1.27
Show that the addition of integers is commutative.

Task 1.28
Show that the addition of integers is associative.

Task 1.29
Show that the addition of integers has $(0,0)_\sim$ as its neutral element.

Task 1.30
Show that for all $a, b \in \mathbb{N}$ the following holds: $(a,b)_\sim + (b,a)_\sim = (0,0)_\sim$. Thus every element in $\mathbb{Z}$ has an additive inverse element.

Task 1.31

1. The map $\phi : \mathbb{N} \to \mathbb{Z}$ defined by $\phi(n) = (n, 0)_\sim$ is injective.

2. For all $m, n \in \mathbb{N}$ the following holds: $\phi(m) + \phi(n) = \phi(m + n)$.

From now on we will **identify** $\mathbb{N}$ with $\phi(\mathbb{N})$.

Task 1.32

1. Define integer multiplication and show that the multiplication is well-defined.

2. Show that $1 = (1, 0)_\sim$ is the neutral element with respect to multiplication.

It is not hard to show that multiplication is commutative and associative. Moreover the distributive law holds in $\mathbb{Z}$.

Task 1.33

With ϕ as defined in Task 1.31, show that

$$\phi(m) \cdot \phi(n) = \phi(m \cdot n).$$

Last not least we will define a relation on $\mathbb{Z}$ as follows:

$$m \leq n \text{ if and only if } n + (-m) \in \mathbb{N}.$$

Task 1.34
Let $a, b, c, d \in \mathbb{N}$. Then $(a,b)_\sim \leq (c,d)_\sim$ if and only if there is a $k \in \mathbb{N}$ such that

$$(a+k, b) \sim (c,d).$$

The next two tasks show that "$\leq$" is a **total order** on $\mathbb{Z}$:

Task 1.35
Show that "$\leq$" is reflexive, anti-symmetric and transitive on $\mathbb{Z}$.

Task 1.36
$m \leq n$ or $n \leq m$ for all $m, n \in \mathbb{Z}$.

Task 1.37
If $m \leq n$, then $m + k \leq n + k$ for all $k \in \mathbb{Z}$.

Task 1.38
If $m \leq n$ and $0 \leq k$, then $m \cdot k \leq n \cdot k$.

1.3 The Rational Numbers

Once again we define the next larger set as certain equivalence classes. On $\mathbb{Z}\times\mathbb{Z}\setminus\{0\}$, we define a relation $\cong$ as follows:

$$(a,b) \cong (c,d) \text{ if and only if } a\cdot d = b\cdot c.$$

We write equivalence classes in the familiar way

$$\frac{a}{b} = \{(c,d) \mid (c,d) \cong (a,b)\},$$

and denote the rational numbers by

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{Z}\setminus\{0\} \right\}.$$

For integers n we write n instead of $\frac{n}{1}$.

We define an order on $\mathbb{Q}$ as follows:

$$0 \leq \frac{a}{b} \text{ if and only if } (0 \leq a \text{ and } 0 < b) \text{ or } (a \leq 0 \text{ and } b < 0).$$

For $p,q \in \mathbb{Q}$, we write $p \leq q$ if $0 \leq q - p$.

With the natural addition and multiplication

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \text{and} \quad \frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$$

and the order above, the set of rational numbers becomes an **ordered field**:

Theorem. $(\mathbb{Q},+,\cdot,\leq)$ has the following properties:

1. $(\mathbb{Q},+)$ is an Abelian group with neutral element 0.
2. $(\mathbb{Q}\setminus\{0\},\cdot)$ is an Abelian group with neutral element 1.
3. $(a+b)\cdot c = a\cdot c + b\cdot c$.
4. $(\mathbb{Q},\leq)$ is a total order.
5. (a) $a \leq b$ implies $a+c \leq b+c$ for all $a,b,c \in \mathbb{Q}$.

(b) $a \leq b$ implies $a \cdot c \leq b \cdot c$ for all $a, b, c \in \mathbb{Q}$ with $0 \leq c$.

Let us write $a < b$ if $a \leq b$ and $a \neq b$. We will say that a is *positive*, if $0 < a$. Similarly, a is called *negative*, if $0 < -a$.

Task 1.39
Let $a, b \in \mathbb{Q}$, and assume $a > b$ and $b > 0$. Then $a^2 > b^2$.

Task 1.40
$\mathbb{Q}$ is *dense in itself*: For all $a, b \in \mathbb{Q}$ with $a < b$ there is a $c \in \mathbb{Q}$ with $a < c < b$.

1.4 The Real Numbers

Completeness. While the rational numbers have nice algebraic properties with respect to their addition, their multiplication and their order, they have one crucial deficiency: The set of rational numbers has "holes".

For instance, the increasing sequence of rational numbers

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

approaches the non-rational number $\sqrt{2}$, a fact well known since antiquity.

We want to remedy this deficiency by constructing an ordered field F containing the rational numbers, which is "complete" in the following sense:

> **(C1)** Every increasing bounded sequence of elements in F converges to an element in F.[13]

Calculus books usually introduce completeness of the set of real numbers in this fashion.

It is convenient to describe completeness also in a different way.

We say a **non-empty** set $A \subseteq F$ is *bounded from above*, if there is a $b \in F$ such that $a \leq b$ for all $a \in A$. Such an element b is then called an upper bound for the set A.

If $A \subseteq F$ is bounded from above, we say that A has a *least upper bound*, denoted by $\sup(A) \in F$, if

1. $\sup(A)$ is an upper bound of A, and
2. for all upper bounds b of A, we have $\sup(A) \leq b$.

Note that $\sup(A)$ must be in F, but we do not require that $\sup(A)$ is an element of A.

[13] A *sequence* is a function $\phi : \mathbb{N} \to F$.
A sequence $\phi : \mathbb{N} \to F$ is called *increasing*, if $m \leq n$ implies $\phi(m) \leq \phi(n)$.
An increasing sequence $\phi : \mathbb{N} \to F$ is called *bounded*, if there is a $b \in F$ such that $\phi(n) \leq b$ for all $n \in \mathbb{N}$.
We say that the increasing sequence ϕ *converges* to $a \in F$, if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $a - \varepsilon \leq \phi(n) \leq a$ for all $n \geq N$.

Task 1.41
Let $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$. Show that A is bounded from above, but fails to have a least upper bound in $\mathbb{Q}$.

The *greatest lower bound* of a set is defined analogously:

We say a non-empty set $A \subseteq F$ is *bounded from below*, if there is a $b \in F$ such that $b \leq a$ for all $a \in A$. Such an element b is then called a lower bound for the set A.

If $A \subseteq F$ is bounded from below, we say that A has a *greatest lower bound*, denoted by $\inf(A) \in F$, if

1. $\inf(A)$ is a lower bound of A, and
2. for all lower bounds b of A, we have $b \leq \inf(A)$.

Task 1.42
Show the following are equivalent:

1. All subsets of F that are bounded from above have a least upper bound.
2. All subsets of F that are bounded from below have a greatest lower bound.

Completeness can then be stated as follows:

(C2) Every subset A of F, which is bounded from above, has a least upper bound.

Task 1.43
Show that property **(C2)** implies property **(C1)**.

Task 1.44
Show that property **(C1)** implies property **(C2)**.

Constructions of the real numbers. Historically, three "constructions" of the real numbers gained prominence in the 19th century, due to Richard Dedekind (Dedekind cuts), Georg Cantor and Augustin-Louis Cauchy (fundamental sequences), and Paul Bachmann (nested intervals), respectively. We will present the first construction below.

Dedekind Cuts. Given two sets of rational numbers $\emptyset \neq L, U \subseteq \mathbb{Q}$, we say that (L, U) is a *partition* of $\mathbb{Q}$ (into two sets), if $L \cup U = \mathbb{Q}$ and $L \cap U = \emptyset$.

A partition (L, U) of $\mathbb{Q}$ is called a *Dedekind cut*, if the following properties hold:

1. If $a \in L$ and $b \in U$, then $a < b$.
2. U has no minimal element.

Here, the element x of a non-empty set A of rational numbers is called *minimal element* of A, if $x \leq a$ for all $a \in A$.

L and U are complementary sets: $U = \mathbb{Q} \setminus L$, and $L = \mathbb{Q} \setminus U$.

We say that two Dedekind cuts (L_1, U_1) and (L_2, U_2) are *equal* and write $(L_1, U_1) = (L_2, U_2)$, if $U_1 = U_2$ (or equivalently, $L_1 = L_2$).

Here are two examples of Dedekind cuts:

Task 1.45
Show that
$$L = \{q \in \mathbb{Q} \mid q \leq -3\},\ U = \{q \in \mathbb{Q} \mid q > -3\}$$
defines a Dedekind cut.

The two sets above “meet” at the rational number -3.

Task 1.46
Show that
$$L = \{q \in \mathbb{Q} \mid q \leq 0 \text{ or } q^2 < 2\},\ U = \{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 > 2\}$$
defines a Dedekind cut.

Here the two sets of the Dedekind cut “meet” at the irrational number $\sqrt{2}$.

Dedekind then defined the set of real numbers to be the set of all Dedekind cuts:
$$\mathbb{R} = \{(L, U) \mid (L, U) \text{ is a Dedekind cut}\}.$$

Note that the rational number $q \in \mathbb{Q}$ corresponds to the Dedekind cut, defined by $L = (-\infty, q] \cap \mathbb{Q}$, $U = (q, \infty) \cap \mathbb{Q}$. We will denote this Dedekind cut by $\underline{q}$.

Addition of Dedekind cuts. Given two Dedekind cuts (L_1, U_1) and (L_2, U_2) we define their sum to be the Dedekind cut (X, Y), where
$$Y = \{y \in \mathbb{Q} \mid y = u_1 + u_2 \text{ for some } u_1 \in U_1 \text{ and } u_2 \in U_2\},$$
$$\text{and } X = \mathbb{Q} \setminus Y.$$

Task 1.47
Show that (X, Y) is indeed a Dedekind cut.

Task 1.48
Let $p, q \in \mathbb{Q}$. Show: $\underline{p} + \underline{q} = \underline{p+q}$.

Task 1.49
Show that the Dedekind cuts with the addition defined above form an Abelian group (see p. 12). What is the neutral element? What is the additive inverse of a Dedekind cut?

Note that the previous task makes it, in particular, possible to define the difference of two Dedekind cuts.

Next, we can define an order on Dedekind cuts: We say that $(L_1, U_1) \leq (L_2, U_2)$, if $L_1 \subseteq L_2$. In particular, (L, U) is non-negative, if $(-\infty, 0] \cap \mathbb{Q} \subseteq L$. We say $(L_1, U_1) < (L_2, U_2)$, if $(L_1, U_1) \leq (L_2, U_2)$ and $(L_1, U_1) \neq (L_2, U_2)$

Clearly $\leq$ is reflexive, anti-symmetric and transitive (why?). The order is also total:

Task 1.50
For any two Dedekind cuts (L_1, U_1) and (L_2, U_2),

$$(L_1, U_1) \leq (L_2, U_2) \text{ or } (L_2, U_2) \leq (L_1, U_1).$$

It is harder to define the multiplication of Dedekind cuts. If both (L_1, U_1) and (L_2, U_2) are non-negative, we define their product (X, Y) by setting

$$Y = \{y \in \mathbb{Q} \mid y = u_1 \cdot u_2 \text{ for some } u_1 \in U_1 \text{ and } u_2 \in U_2\},$$

$$\text{and } X = \mathbb{Q} \setminus Y.$$

Task 1.51
Check that the product defined above is indeed a Dedekind cut.

To define the product of arbitrary Dedekind cuts, one first needs the following result:

Theorem. Every Dedekind cut is the difference of two non-negative Dedekind cuts.

The product of two arbitrary Dedekind cuts is then defined by "multiplying out"; the concept is well-defined.

With these definitions one can show with quite a bit more work:

Theorem. The real numbers with the addition, multiplication and order defined above form an **ordered field**.

The Dedekind cut $\underline{1} := (\mathbb{Q} \cap (-\infty, 1], \mathbb{Q} \cap (1, \infty))$ is the neutral element with respect to multiplication. The existence of a multiplicative inverse is first shown for positive Dedekind cuts, and then generalized to negative Dedekind cuts.

Completeness of Dedekind cuts. Note that a Dedekind cut (L', U') is an upper bound for a set of Dedekind cuts $\mathcal{D}$, if $L \subseteq L'$ for all $(L, U) \in \mathcal{D}$.

Task 1.52
Let
$$\mathcal{D} = \left\{ \left(\mathbb{Q} \cap (-\infty, -\frac{1}{n}], \mathbb{Q} \cap (-\frac{1}{n}, \infty) \right) \mid n \in \mathbb{N} \right\}.$$
Show that $\mathcal{D}$ is bounded from above, then determine its least upper bound.

Finally we can show that the set of real numbers defined via Dedekind cuts is **complete**:

Task 1.53
Show that $\mathbb{R}$, the set of all Dedekind cuts, satisfies Axiom **(C2)**.

Task 1.54
Show that $\mathbb{Q}$ is dense in $\mathbb{R}$: Given two Dedekind cuts $(L_1, U_1) < (L_2, U_2)$, there is a $q \in \mathbb{Q}$ such that
$$(L_1, U_1) \leq \underline{q} \leq (L_2, U_2).$$

2 Numerical Series

From this section onward the standard results from a first Analysis course are a prerequisite. For this section in particular you can (and will need to) use results about numerical sequences.

Given a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, the *infinite series* $\displaystyle\sum_{n=0}^{\infty} a_n$ is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

The corresponding *sequence of partial sums* $(s_k)_{k\in\mathbb{N}}$ is defined by

$$s_k = a_1 + a_2 + a_3 + \cdots a_k.$$

If the sequence of partial sums converges, with limit s, we say that the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges, and we write

$$\sum_{n=0}^{\infty} a_n = s.$$

We will often write $\sum a_n$ instead of $\displaystyle\sum_{n=0}^{\infty} a_n$. Sometimes the summation will not start at $n = 0$.

Task 2.1
Show that the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges if and only if there is a $k \in \mathbb{N}$ such that $\displaystyle\sum_{n=k}^{\infty} a_n$ converges.

This task does not imply that for a given $k \neq 0$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=k}^{\infty} a_n.$$

The following are also direct consequences of the corresponding facts for sequences:

1. If $b \in \mathbb{R}$ and the series $\sum a_n$ converges, then the sum $\sum (b \cdot a_n)$ converges as well, and
$$\sum_{n=0}^{\infty} (b \cdot a_n) = b \cdot \sum_{n=0}^{\infty} a_n.$$
2. If the series $\sum a_n$ and $\sum b_n$ both converge, then their sum $\sum (a_n + b_n)$ converges as well, and
$$\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n.$$

Task 2.2
If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ converges if and only if the corresponding sequence of partial sums (s_k) is bounded.

Task 2.3
Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Hint: Show that the partial sums satisfy $s_k \leq 2 - \dfrac{1}{k}$.

This implies that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$. Euler showed that the limit is actually equal to $\dfrac{\pi^2}{6} \approx 1.64493$.

Task 2.4
Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (= does not converge).

Hint: Show that the partial sums satisfy $s_{2^k} \geq 1 + \frac{k}{2}$.

Task 2.5
The series $\sum a_n$ converges if and only if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that whenever $m > n \geq N$ it follows that

$$|a_{n+1} + a_{n+2} + \cdots a_m| < \varepsilon.$$

Task 2.6
If $\sum a_n$ converges, then (a_n) converges to 0.

Note that by the example in Task 2.4 the converse of Task 2.6 does **not** hold.

Task 2.7
Show: If the series $\sum_{n=0}^{\infty} |a_n|$ converges, so does $\sum_{n=0}^{\infty} a_n$.

If $\sum_{n=0}^{\infty} |a_n|$ converges, we say that $\sum_{n=0}^{\infty} a_n$ *converges absolutely.* If on the other hand, $\sum_{n=0}^{\infty} a_n$ converges while $\sum_{n=0}^{\infty} |a_n|$ diverges, we say that $\sum_{n=0}^{\infty} a_n$ *converges conditionally.*

Task 2.8
Show that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges conditionally.

The example above is a special case of the next task:

Task 2.9
Suppose the sequence (a_n) satisfies

1. $a_0 \geq a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, and
2. the sequence (a_n) converges to 0,

then $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

Given a series $\displaystyle\sum_{n=0}^{\infty} a_n$, we say the series $\displaystyle\sum_{n=0}^{\infty} b_n$ is a *rearrangement* of $\displaystyle\sum_{n=0}^{\infty} a_n$, if there is a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that $b_{\varphi(n)} = a_n$ for all $n \in \mathbb{N}$.

Task 2.10
If the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges absolutely, then any rearrangement of $\displaystyle\sum_{n=0}^{\infty} a_n$ converges to the same limit.

In other words: If a series is absolutely convergent, then it is "infinitely commutative." If, on the other hand, a series converges only conditionally, then commutativity fails in a spectacular way:

Task 2.11
Suppose that the series $\sum_{n=0}^{\infty} a_n$ converges conditionally. Then for every $s \in \mathbb{R}$, there is a rearrangement $\sum_{n=0}^{\infty} b_n$ of $\sum_{n=0}^{\infty} a_n$ such that $\sum_{n=0}^{\infty} b_n$ converges to s.

Here are two hints to get you started on this problem:

1. Let $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = \max\{-a_n, 0\}$. Thus $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$. Observe that both series $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ do not converge. Therefore both partial sums are not bounded.
2. The series in Task 2.8 actually converges to $\ln 2 \approx 0.693147$. Can you find a recipe how to rearrange the series so that the rearrangement converges to 1 instead?

3 Sequences and Series of Functions

3.1 Pointwise and Uniform Convergence

We now turn our attention to the convergence of sequences and series of functions. Here is the natural definition to extend the notion of convergence from numbers to functions:

Let $D \subseteq \mathbb{R}$. Given functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, we say the sequence $(f_n)_{n\in\mathbb{N}}$ *converges to f pointwise* if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in D$.

Equivalently, this means that for all $x \in D$ and for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ it follows that $|f_n(x) - f(x)| < \varepsilon$.

Task 3.1
Let $f_n : [0,1] \to \mathbb{R}$ be given by $f(x) = x^n$. Find a suitable $f : [0,1] \to \mathbb{R}$ such that (f_n) converges to f pointwise.

This example reveals the first deficiency of pointwise convergence: the pointwise limit of a sequence of continuous functions is not necessarily continuous.

The next example shows that the pointwise limit of a sequence of bounded functions is not necessarily bounded. We say a function $f : D \to \mathbb{R}$ is *bounded* if there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in D$.

Task 3.2
Let $f_n : (-1,1) \to \mathbb{R}$ be given by $f_n(x) = \sum_{k=0}^{n} x^k$. Show that (f_n) converges pointwise to the function $f(x) = \dfrac{1}{1-x}$.

Let us say that a sequence $f_n : D \to \mathbb{R}$ is *uniformly bounded*, if there is an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in D$ and $n \in \mathbb{N}$.

Assuming this extra assumption, we obtain a positive result:

Task 3.3
Suppose the sequence $f_n : D \to \mathbb{R}$ is uniformly bounded and converges pointwise to the function f. Then f is bounded.

Pointwise convergence also does not interact nicely with Riemann integration:

Task 3.4
Let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 0, \text{ if } x = 0 \\ n, \text{ if } 0 < x \leq \frac{1}{n} \\ 0, \text{if } \frac{1}{n} < x \leq 1 \end{cases}$$

Show that this sequence converges pointwise to the zero-function.

Observe that $\displaystyle\int_0^1 f_n(x)\,dx = 1$ for all n, while the pointwise limit has integral 0.

We have already seen in Task 3.1 that the pointwise limit of differentiable functions is not necessarily differentiable. The next example shows that even in the case when the pointwise limit is differentiable, its derivative does not necessarily have the desired properties.

Task 3.5
Let $f_n : [-1,1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x}{1+nx^2}$. Show that each f_n is differentiable, that (f_n) has as its pointwise limit f the zero function, but that $\lim\limits_{n\to\infty} f_n'(0) \neq f'(0)$.

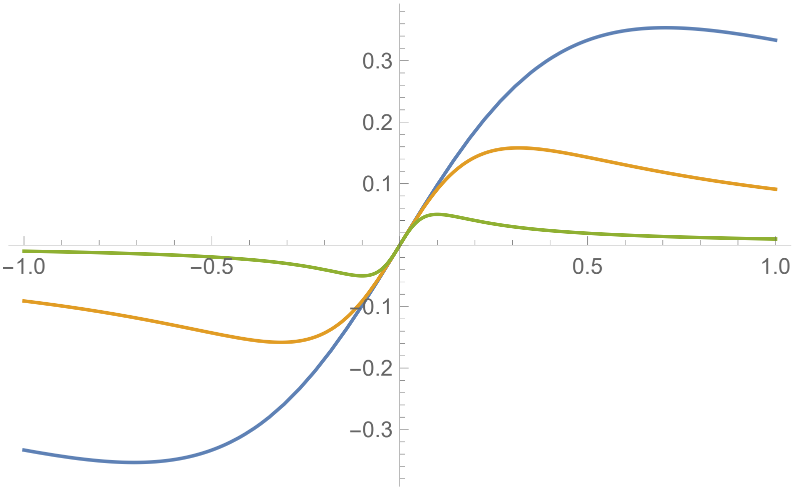

Figure 1: The functions f_2, f_{10} and f_{100} from Task 3.5

All these examples show that pointwise convergence is not such a useful property[14].

We will therefore study a different limit concept for functions:

Let $D \subseteq \mathbb{R}$. Given functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, we say the sequence $(f_n)_{n\in\mathbb{N}}$ *converges to f uniformly* if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in D$ it follows that $|f_n(x) - f(x)| < \varepsilon$.

Task 3.6
Let functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$ be given. If (f_n) converges to f uniformly, then (f_n) converges to f pointwise.

Task 3.7
Show that the converse of Task 3.6 is false.

[14]Actually, in the case of integration, this led to the development of a different notion of integration: the Lebesgue integral with its Dominated Convergence Theorem.

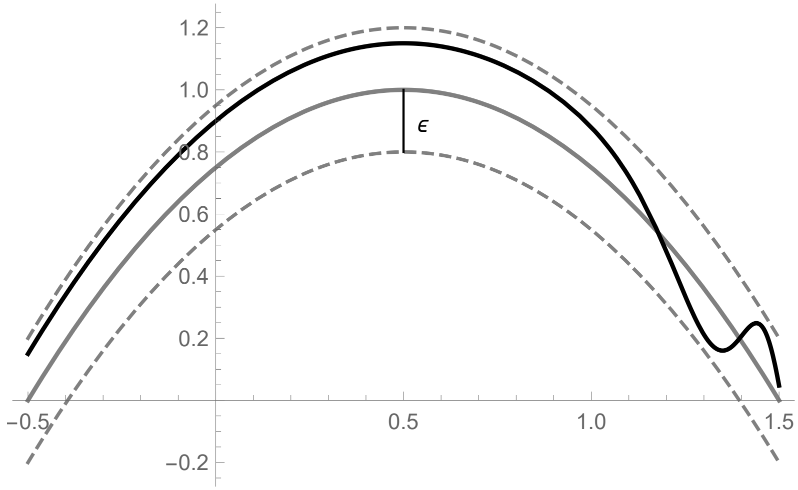

Figure 2: Uniform convergence: The function f_n (in black) lies in an ε-tube around the function f (in gray).

Task 3.8
Let a sequence of functions $f_n : D \to \mathbb{R}$ be given. The sequence (f_n) converges uniformly if and only if for all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that for all $x \in D$ and for all $m, n \geq N$ it follows that $|f_m(x) - f_n(x)| < \varepsilon$.

Task 3.9
Let a sequence of functions $f_n : D \to \mathbb{R}$ and numbers $M_n \geq 0$ be given. Suppose

$$|f_n(x)| \leq M_n \text{ for all } x \in D \text{ and } n \in \mathbb{N}.$$

Show: If $\sum M_n$ converges, then $\sum f_n$ converges uniformly (and absolutely).

The next two results highlight some of the permanence properties of uniform convergence:

Task 3.10
Let $f_n : D \to \mathbb{R}$ be continuous functions such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some function f. Then f is continuous.

Task 3.11
Let $f_n : D \to \mathbb{R}$ be bounded functions such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some function f. Then f is bounded.

There are two more permanence results for uniform convergence:

Theorem 3.1. Let $f_n : [a,b] \to \mathbb{R}$ be Riemann-integrable functions such that (f_n) converges uniformly to some function f. For $t \in [a,b]$, let $F_n(t) = \displaystyle\int_a^t f_n(x)\,dx$.

Then f is Riemann integrable, and moreover (F_n) converges uniformly to the function F, defined by $F(t) = \displaystyle\int_a^t f(x)\,dx$.

Theorem 3.2. Let $f_n : [a,b] \to \mathbb{R}$ be differentiable functions such that (f_n') converges uniformly to some function g. Assume additionally that for some $x_0 \in [a,b]$ the sequence $(f_n(x_0))$ converges.

Then (f_n) converges uniformly to some function f, f is differentiable on $[a,b]$, and $f'(x) = g(x)$ for all $x \in [a,b]$.